Step 3. $\forall \sigma \neq id$: $\sigma(a)\sigma(b) < 0$

Know: $\Delta(H_{\mathcal{O}})$ irred lattice in $\prod_{\sigma \in S^\infty} H^\sigma$ and proj. in H^{id} lattice

$\Rightarrow$ H^σ cpt $\forall \sigma \neq id$ $\rightarrow$ $H^\sigma \simeq SO(2,1)$ $\forall \sigma \neq id$

$\Rightarrow$ $\sigma(a), \sigma(b), \sigma(-1)$ have the same sign $\Rightarrow$ $\sigma(a), \sigma(b) < 0$.

Prop. $SL(2,\mathbb{Z})$ is the only non-cocpt arith subgp of $SL(2,\mathbb{R})$ up to conjugation and commensurability.

Pf: Work with $SO(2,1)$. Goal: prove that $SO(2,1)_{\mathbb{Z}}$ is the "only" non-cpt arith subgp of $SO(2,1)$.

Know: $\exists G < SL(\ell, \mathbb{R})$ simple for some ℓ s.t. G is defined over some number field F and $\exists \varphi: G \longrightarrow SO(2,1)$ isogeny s.t. $\varphi(G_{\mathcal{O}_F})$ is comm to Γ.

Pf of prev. Prop. $\Rightarrow$ $G = SO(B, \mathbb{R})$ where $B: \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ bilinear form of signature $(2,1)$.

NTS: G is defined over $\mathbb{Q}$.

Recall from the original def of arith gps: $\exists K'$ cpt s.t.

$G' = G \times K' < SL(\ell, \mathbb{R})$, G' defined over $\mathbb{Q}$, $\Gamma K' = G'_{\mathbb{Z}} K'$

N = Zariski closure of unipotent elts in $G'_{\mathbb{Z}}$

K' cpt. $\Rightarrow$ $N \subseteq G$

N is normalised by $G'_{\mathbb{Z}}$ $\xrightarrow[\text{density}]{\text{Borel}}$ $N \triangleleft G$ $\overset{G \text{ simple}}{\Longrightarrow}$ $N = G$.

G is the Zariski closure of a group defined over $\mathbb{Z}$ $\Rightarrow$ G is def. $/\mathbb{Q}$.

Step 2. Have: $G = SO(B, \mathbb{R})$. Want: $B(x,x) = x_1^2 + x_2^2 - x_3^2$

Γ not cocpt $\Rightarrow$ B is isotropic over F $\Rightarrow$ $\exists u \in F^3 \setminus \{0\}$: $B(u,u) = 0$.

Choose $v \in F^3 \setminus \{0\}$ s.t. $B(u,v) \neq 0$

By adding a multiple of u wma $B(v,v) = 0$.

Choose $w \in F^3 \setminus \{0\}$ s.t. $B(w,u) = B(w,v) = 0$.

Multiplying B with scalars preserves everything $\Rightarrow$ wma $B(w,w) = 2B(u,v) = 1$.

$\rightarrow$ B has the desired form wrt. the basis $\{w, u+v, u-v\}$.

This week we again describe the arithmetic subgroups of $SL(2,\mathbb{R})$ via quaternion algebras.

Def. Let F be a field, $a,b \in F^\times$. Then the corresponding <u>quaternion algebra</u> is the ring $\underline{\mathbb{H}_F^{a,b}} = \{q + pi + rj + sk \mid p,q,r,s \in F\}$

with addition coming from F and multiplication determined by

$$i^2 = a, \quad j^2 = b, \quad ij = -ji = k$$

with F lying in the center of $\mathbb{H}_F^{a,b}$.

Def. The <u>reduced norm</u> $\underline{N_{red}}: \mathbb{H}_F^{a,b} \longrightarrow F$ is defined as

$$N_{red}(p+qi+rj+sk) = (p+qi+rj+sk)\cdot\overline{(p+qi+rj+sk)} = p^2 - aq^2 - br^2 + abs$$

where the <u>conjugation</u> is defined as $\overline{p+qi+rj+sk} = p - qi - rj - sk$.

Note that $\overline{x\cdot y} = \overline{y}\cdot\overline{x}$. and $\mathbb{H}_F^{a,b} \simeq \mathbb{H}_F^{b,a}$ (flipping i and j).

Ex. 1) $\mathbb{H}_{\mathbb{R}}^{-1,-1} = \mathbb{H}$ Hamiltonian quaternions

2) $\mathbb{H}_F^{t^2a,t^2b} \simeq \mathbb{H}_F^{a,b} \quad \forall a,b,t \in F^\times$ (Exc.)

3) $\mathbb{H}_F^{a^2,b} \simeq Mat_{2\times2}(F) \quad \forall a,b \in F^\times$

The isomorphism $\varphi: \mathbb{H}_F^{a^2,b} \longrightarrow Mat_{2\times2}(F)$ is given by

$$\varphi(1) = id, \quad \varphi(i) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad \varphi(j) = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}, \quad \varphi(k) = \begin{pmatrix} 0 & a \\ -ba & 0 \end{pmatrix}.$$

<u>Lemma.</u> $\mathbb{H}_{\mathbb{C}}^{a,b} \simeq Mat_{2\times2}(\mathbb{C}) \quad \forall a,b \in \mathbb{C}^\times$ and $\mathbb{H}_{\mathbb{R}}^{a,-1} \simeq \begin{cases} Mat_{2\times2}(\mathbb{R}) & a>0 \\ \mathbb{H} & a<0 \end{cases}$

Pf: Follows from 3). Exc: show $Mat_{2\times2}(\mathbb{R}) \not\simeq \mathbb{H}$. $\square$

<u>Prop.</u> $a,b>0$ integers. $\underline{SL(1,\mathbb{H}_{\mathbb{R}}^{a,b})} := \{g \in \mathbb{H}_{\mathbb{R}}^{a,b} \mid N_{red}(g) = 1\}$

Then 1) $SL(1,\mathbb{H}_{\mathbb{R}}^{a,b}) \simeq SL(2,\mathbb{R})$

2) $SL(1,\mathbb{H}_{\mathbb{R}}^{a,b})_{\mathbb{Z}} < SL(1,\mathbb{H}_{\mathbb{R}}^{a,b})$ is an arith subgp.

where $SL(1,\mathbb{H}_{\mathbb{R}}^{a,b})_{\mathbb{Z}} = \{x_0 + x_1 i + x_2 j + x_3 k \in SL(1,\mathbb{H}_{\mathbb{R}}^{a,b}) \mid \forall x_i \in \mathbb{Z}\}$.

3) TFAE:

(a) $SL(1,\mathbb{H}_{\mathbb{R}}^{a,b})_{\mathbb{Z}} < SL(1,\mathbb{H}_{\mathbb{R}}^{a,b})$ is cocompact

(b) $(0,0,0,0)$ is the only integral solution of $w^2 - ax^2 - by^2 + abz^2 = 0$

(c) Every $\neq 0$ elt in $\mathbb{H}_{\mathbb{Q}}^{a,b}$ has a multiplicative inverse, i.e. $\mathbb{H}_{\mathbb{Q}}^{a,b}$ is a division algebra.

Pf: 1) Know: $\mathbb{H}^{a,b}_{\mathbb{R}} \simeq \mathrm{Mat}_{2\times 2}(\mathbb{R})$ via $\varphi(1) = \mathrm{id}$, $\varphi(i) = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}$,

$\varphi(j) = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$, $\varphi(k) = \begin{pmatrix} 0 & \sqrt{a} \\ -b\sqrt{a} & 0 \end{pmatrix}$.

NTS: $N_{red} = 1 \iff \det = 1$, i.e. $N_{red}(g) = \det(\varphi(g))$.

$g := p + qi + rj + sk$

$$\det \varphi(g) = \det \begin{pmatrix} p+q\sqrt{a} & r+s\sqrt{a} \\ br - bs\sqrt{a} & p - q\sqrt{a} \end{pmatrix} = p^2 - aq^2 - b(r^2 - as^2) = N_{red}(g). \checkmark$$

2) We could use the Prop. from last week and just go through the list.

Or directly: $SL(1, \mathbb{H}^{a,b}_{\mathbb{R}}) \curvearrowright \mathbb{H}^{a,b}_{\mathbb{R}}$,

$\mathbb{H}^{a,b}_{\mathbb{Q}}$ is a $\mathbb{Q}$-form of $\mathbb{H}^{a,b}_{\mathbb{R}}$

$\mathbb{H}^{a,b}_{\mathbb{Z}}$ is a $\mathbb{Z}$-lattice of the $\mathbb{Q}$-form $\mathbb{H}^{a,b}_{\mathbb{Q}}$

and $\gamma(\mathbb{H}^{a,b}_{\mathbb{Z}}) \subseteq \mathbb{H}^{a,b}_{\mathbb{Z}}$ iff $\gamma \in \mathbb{H}^{a,b}_{\mathbb{Z}}$

$\Rightarrow SL(\mathbb{H}^{a,b}_{\mathbb{R}}) \cap \mathbb{H}^{a,b}_{\mathbb{Z}} = SL(\mathbb{H}^{a,b}_{\mathbb{R}})_{\mathbb{Z}}$ is an arithmetic subgp. $\checkmark$

3) (c) $\Rightarrow$ (a): We will show that $G_{\mathbb{Z}}$ non cocpt $\Rightarrow \mathbb{H}^{a,b}_{\mathbb{Q}}$ not a div. alg.

Here $G = SL(1, \mathbb{H}^{a,b}_{\mathbb{R}})$.

Godement $\Rightarrow G_{\mathbb{Z}}$ has a nontrivial unipotent element

$\exists \gamma \in G_{\mathbb{Z}} \setminus \{1\}$ that has 1 as an eigenvalue, i.e.

$\exists v \in \mathbb{H}^{a,b}_{\mathbb{Q}} \setminus \{0\}$ s.t. $\gamma v = v \Rightarrow (\gamma - 1)v = 0 \Rightarrow (\gamma - 1) \in \mathbb{H}^{a,b}_{\mathbb{Q}} \setminus \{0\}$ is

a zero divisor. $\checkmark$

(a) $\Rightarrow$ (c): We will prove: if $\mathbb{H}^{a,b}_{\mathbb{Q}}$ is not a div. alg $\Rightarrow G_{\mathbb{Z}}$ not cocpt.

Wedderburn's Theorem (PO): if $\mathbb{H}^{a,b}_{\mathbb{Q}}$ is not a div. alg. $\Rightarrow \mathbb{H}^{a,b}_{\mathbb{Q}} \simeq \mathrm{Mat}_{2\times 2}(\mathbb{Q})$.

(Note that WT is more general, this is just a special case.
For a direct pf, cf. Exc. 6.2.3.)

$\Rightarrow SL(1, \mathbb{H}^{a,b}_{\mathbb{Z}}) \simeq SL(2, \mathbb{Z})$, which is not cocpt.

(b) $\Rightarrow$ (c) Exc. 6.2.4.

(a) $\Rightarrow$ (b) already shown (where?)

The following can be proven in a similar fashion:

<u>Prop.</u>
- $F \neq \mathbb{Q}$ totally real number field
- $\mathcal{O}_F$ ring of integers
- $a, b \in F^+$, $\sigma(a), \sigma(b) < 0 \quad \forall \sigma \neq id$
- $G = SL(1, \mathbb{H}^{a,b}_{\mathbb{R}}) \simeq SL(2, \mathbb{R})$

$\Rightarrow G_{\mathcal{O}_F} = SL(1, \mathbb{H}^{a,b}_{\mathcal{O}_F})$ cocpt arith subgp of G. $\square$

<u>Prop.</u> Every cocpt. arith subgp. of $SL(2,\mathbb{R})$ is of one of the two forms above, up to conjugation and commensurability.

Pf: Recall: the two types are

(1) $a, b > 0$ integers, $(0,0,0)$ is the sole integral solution to $ax^2 + by^2 - z^2 = 0$. Then for $G = SO(ax^2 + by^2 - z^2; \mathbb{R})$ we have that $G_{\mathbb{Z}} < G$ is cocpt.

(2) $F \neq \mathbb{Q}$ totally real number field, $a, b \in F^+$, $\sigma(a), \sigma(b) < 0 \quad \forall \sigma \neq id$. Then $G_{\mathcal{O}_F} < G$ is a cocpt. arith. subgp.

We will show that (1) is of the form $SL(1, \mathbb{H}^{a,b}_{\mathbb{Z}})$.

<u>Claim 1</u>. If $\exists g \in \mathbb{H}^{a,b}_F \setminus \{0\}$ s.t. $N_{red}(g) = 0$ then $\exists g' \in \mathbb{H}^{a,b}_F \setminus \{0\}$ that has k-component 0 and $N_{red}(g') = 0$

Pf: $g = x_0 + ix_1 + jx_2 + kx_3$, $x_3 \neq 0$, $\alpha := \frac{-x_2}{x_3} + i$

$\alpha g \neq 0$ but its k-cponent is 0.

$N_{red}(\alpha g) = 0$ $\boxed{\text{c}}$

$\rightarrow \exists (w, x, y, z) \in \mathbb{Q}^4 \setminus \{0\}$ with $N_{red}(w + xi + yj + zk) = 0$

iff $\exists (x, y, z) \in \mathbb{Q}^3 \setminus \{0\}$ with $ax^2 + by^2 = z^2$.

So if Γ is of the form (1) $\Rightarrow G_{\mathbb{Z}} = SL(1, \mathbb{H}^{a,b}_{\mathbb{Z}}) < SL(2,\mathbb{R}) = G$ is a cocpt arith subgp.

Goal: show that Γ and $G_{\mathbb{Z}}$ are commensurable.

$\mathfrak{g} :=$ Lie alg of G. This can be viewed as a subspace of $\mathbb{H}^{a,b}_{\mathbb{R}}$:

$\mathfrak{g} = \{ v \in \mathbb{H}^{a,b}_{\mathbb{R}} \mid Re(v) = 0 \}$. Exc. 6.2.7.

And $Ad_G(g)(v) = gvg^{-1}$.

$N_{red}|_{\mathfrak{g}}$ is a quadratic form on $\mathfrak{g}$ that is invariant under Ad_G.

$N_{red}(v) = N_{red}(xi + yj + zk) = -ax^2 - by^2 + abz^2$ for $v = xi + yj + zk \in \mathfrak{g}$

Change v'bles: $x \mapsto by$, $y \mapsto ax$:

$N_{red}(v) = \underbrace{-ab}(ax^2 + by^2 - z^2)$

multiplication by a scalar does not change the group.

$\Rightarrow$ arithmetic group of type (1) is commensurable to $G_{\mathbb{Z}}$

Identification of the other two types goes similarly.

14.06.2018

Hyperbolic surfaces

Sources: Beardon — The geometry of discrete gps

Buser — The gty and spectra of cpt Riemann surfaces

Katok — Fuchsian gp

Recall: $\mathbb{H}^2 = SO(2,1)^\circ / SO(2) = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$ $\qquad ds^2 = \dfrac{dx^2 + dy^2}{y^2}$

Def. Hyperbolic surface: locally symmetric space with universal cover $\mathbb{H}^2$.

Closed hyperbolic surface: compact + hyperbolic surface (and has no boundary)

Thm. (19^{th} cty) Classification of surfaces.

Let S be a closed orientable surface. Then S is diffeomorphic to a genus g surface (i.e. # of g tori with S^2) for some $g \geq 0$.

Thm. (Gauss – Bonnet) S can be given the structure of a hyperbolic surface iff it has genus ≥ 2.

Fact. For fixed genus $g \geq 2$ there is a $6g-6$ dimensional deformation space of hyperbolic structures on a genus g surface.

(Equivalently, there are a lot of lattices in $\mathrm{Isom}^+(\mathbb{H}^2)$ that are isometric to $\pi_1(S_g)$.)

Def. The <u>systole</u> of a Riemannian manifold M is

$$\underline{\mathrm{sys}}(M) := \inf \{ \text{lengths of closed geodesics in } M \}$$

Fact. On a hyperbolic surface the systole is realised by a closed geodesic.

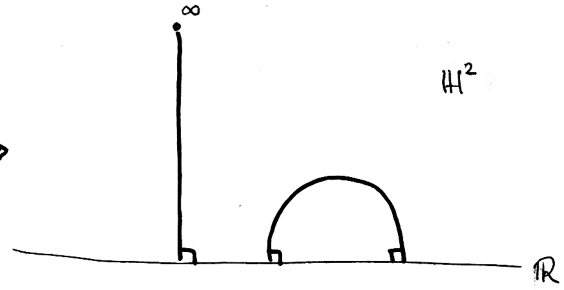
Question. What is $\max \{ \mathrm{sys}(M) \mid M \text{ hyp. surface of genus } g \}$?

Fact. This is OPEN.

Closed geodesics on hyperbolic surfaces

$$\mathbb{H}^2 = \{ z \in \mathbb{C} \mid \mathrm{Im}\, z > 0 \} \qquad ds^2 = \frac{dx^2 + dy^2}{y^2}$$

What are the geodesics in $\mathbb{H}^2$? $\longrightarrow$



Follows from the explicite formula for the metric:
- vertical lines and
- semicircles $\perp \mathbb{R}$

Indeed, any $x, y \in \mathbb{H}^2$ $\exists!$ γ geodesic s.t. $x, y \in \gamma$.

Geodesics are parametrised by pairs of points in $\mathbb{R} \cup \{\infty\} = \partial_\infty \mathbb{H}^2$, i.e. their end points

Isometries: $SL(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \dfrac{az+b}{cz+d}$. This extends to $\partial_\infty \mathbb{H}^2$.

Note: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} z = z \implies$ get an action of $PSL(2,\mathbb{R}) = SL(2,\mathbb{R}) / \pm Id = \mathrm{Isom}^+(\mathbb{H}^2)$

Classification of elements of $PSL(2,\mathbb{R})$

Def. An isometry is called
- <u>hyperbolic</u> if it has two fixed pts on $\partial_\infty \mathbb{H}^2$
- <u>parabolic</u> if it has one fixed pt on $\partial_\infty \mathbb{H}^2$
- <u>elliptic</u> if it has a fixed pt in $\mathbb{H}^2$.

Examples. • Hyperbolic: $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in PSL(2,\mathbb{R})$ for $\lambda > 0$, $z \longmapsto \lambda^2 z$

Fixed pts: $0, \infty$

• Parabolic: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $z \longmapsto z+1$, fixed pt is ∞ only

• Elliptic: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $z \longmapsto \frac{-1}{z}$, has fixed pt i

Fact. • All isometries of $\mathbb{H}^2$ are either hyperbolic, parabolic or elliptic.

• If $g \in PSL(2,\mathbb{R})$ is hyperbolic then $\exists h \in PSL(2,\mathbb{R})$ s.t. $hgh^{-1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ for some $\lambda > 0$

• If $g \in PSL(2,\mathbb{R})$ is parabolic then $\exists h \in PSL(2,\mathbb{R})$ s.t. $hgh^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

• For elliptic isometries, the situation is slightly more complicated and won't be discussed as we won't need them.

Translation distance

Def. Let $g \in PSL(2,\mathbb{R})$ be hyperbolic. Then the translation distance of g is

$$\underline{T_g} := \inf \{ d_{\mathbb{H}^2}(gx, x) \mid x \in \mathbb{H}^2 \}$$

The axis of g is $\underline{\alpha_g} := \{ x \in \mathbb{H}^2 \mid d_{\mathbb{H}^2}(gx, x) = T_g \}$.

Fact. • α_g is the geodesic determined by the 2 fixed pts of g

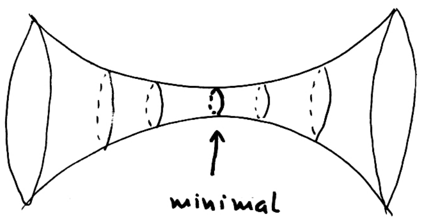
$\Rightarrow g = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ has axis $\alpha_g = \mathbb{R}_{>0}\, i$

• $T_g = 2 \cosh^{-1}\left(\frac{|tr(g)|}{2} \right)$ Note that $tr(g)$ is determined up to sign only, but $|tr(g)|$ is well-defined.

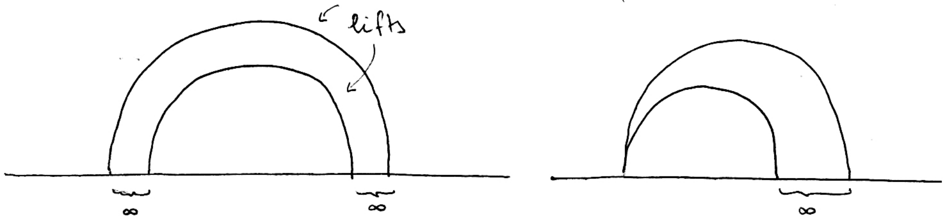
Geodesics on hyperbolic surfaces

Prop. S closed hyperbolic surface $\Rightarrow$ in every free homotopy class of closed curves on S there is a unique geodesic minimising length over the htpy class.

Pf (SKETCH): To find a geodesic, use Arzela-Ascoli argument.



Uniqueness: suppose $\exists$ 2 such geodesics, lift to $\mathbb{H}^2$. Since the original geodesics are of bounded distance from each other, so are their lifts.



If the end pts do not coincide, the distance will not be bounded

$\Rightarrow$ end pts must be the same. But end pts determine the geodesic

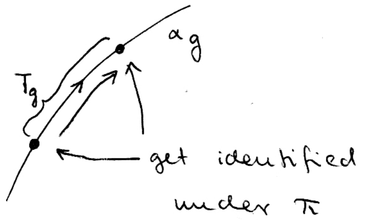
$\Rightarrow$ the lifts must be the same $\Rightarrow$ project to the same geodesic on S. $\square$

<u>Prop.</u> There is a one-to-one correspondence between

$\{$oriented closed geodesics on $S\} \longleftrightarrow \{$conjugacy classes of hyp. elts in $\Gamma\}$

where $S = \Gamma \backslash \mathbb{H}^2$. Moreover if g is in the conjugacy class in Γ corresponding to the geodesic γ then $\ell(\gamma) = T_g$.

PF (SKETCH): $g \in \Gamma$ hyperbolic elt. Then α_g projects to a closed geodesic under the covering map $\pi : \mathbb{H}^2 \twoheadrightarrow \mathbb{H}^2/\Gamma$



$\Rightarrow$ the length of the geodesic is T_g.

The direction in which g translates gives the orientation

E.g. $g = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ translates upwards along α_g if $\lambda > 1$ and downwards if $\lambda < 1$.

Exc.: $\alpha_{hgh^{-1}} = h(\alpha_g)$

$\Rightarrow$ they project to the same geodesic. ✓

The other direction is based on a lifting argument. and $\alpha_{hgh^{-1}} = h(\alpha_g)$. $\square$

<u>Conclusion:</u> to control systole, we need to control the lowest trace of hyperbolic elts in Γ.

Note that if S is closed, every nontrivi elt is hyperbolic:

if elliptic of finite order $\longrightarrow$ cone pt.

of infinite order $\longrightarrow$ problem with proper discontinuity

if parabolic: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is unipotent, hence all parabolics are unipotent (unipotency is conjugacy invariant), problem with cptness.

$$\Rightarrow \quad \mathrm{sys}\left(\Gamma \backslash \mathbb{H}^2\right) = \min\left\{ 2\cosh^{-1}\left(\frac{|\mathrm{tr}(g)|}{2}\right) \;\middle|\; g \in \Gamma \setminus \{0\} \right\}$$

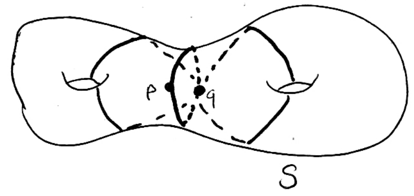
as hyperbolic is equivalent to non-trivial in Γ.

A simple upper bound on $\mathrm{sys}(S)$

Lemma. S closed orientable hyp. surface of genus g.

$$\Rightarrow \quad \mathrm{sys}(S) \leq 2\log(4g-2).$$

Pf:

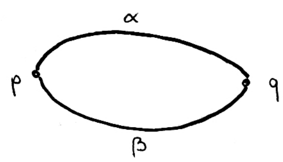


$p \in S$ point

$D_r(p) =$ disk of radius r around p

Suppose $D_r(p)$ is not embedded $\Rightarrow$

then two radii of $D_r(p)$ meet.



Claim: the curve they form is non-contractible.

Pf: $\nexists$ $\rightarrow$ get 2 geodesics in $\mathbb{H}^2$ by lifting α and β which intersect twice. ↯

$\Rightarrow$ we found a closed geodesic of length $\leq 2r$.

$\nexists$ closed geod. of length $\leq \mathrm{sys}(S)$ by definition.

So if $D_r(p)$ is not embedded then $2r \leq \mathrm{sys}(S)$.

In particular, $D_{\frac{\mathrm{sys}(S)}{2}}(p)$ must embed in S.

$$\Rightarrow \quad \mathrm{area}\left(D_{\frac{\mathrm{sys}(S)}{2}}(p)\right) \leq \mathrm{area}(S)$$

Gauss – Bonnet: the Gauss curvature K is -1 on S.

$$-\mathrm{area}(S) = \int_S K = 2\pi\,\chi(S) = 4\pi(1-g)$$

$$\underbrace{\mathrm{area}\left(D_{\frac{\mathrm{sys}(S)}{2}}(p)\right)}_{\substack{\text{isometric to a disc} \\ D_r(\tilde{p}) \text{ in } \mathbb{H}^2}} = \mathrm{area}_{\mathbb{H}^2}\left(D_r(\tilde{p})\right) = 2\pi\left(\cosh(r) - 1\right)$$

where $r = \frac{\mathrm{sys}(S)}{2}$

$$2\pi\left(\cosh\left(\frac{\mathrm{sys}(S)}{2}\right) - 1\right) \leq 4\pi(g-1)$$

The assertion follows.

Next week: (Buser—Sarnak) $\exists$ sequence S_k of arithmetic hyp surfaces of genus g_k s.t. $g_k \to \infty$ as $k \to \infty$ and $sys(S_k) \gtrsim \frac{4}{3}\log(g_k)$.

19.06.2018

Recall the following:

<u>Prop.</u> There are 1-to-1 correspondences

$$\left\{\begin{matrix}\text{closed geodesics}\\ \text{on } S\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{free htp. classes of htpically}\\ \text{nontrivial closed curves on } S\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{conjugacy classes}\\ \text{of } \Gamma\setminus\{e\}\end{matrix}\right\}$$

Moreover for a closed geodesic γ we have $\ell(\gamma) = 2\cosh^{-1}\left(\frac{|tr\,\gamma|}{2}\right)$.

Today we will construct arithmetic surfaces (i.e. $\Gamma\setminus\mathbb{H}^2$ with Γ arithmetic) with large systoles. Reference: "On the period matrix of a Riemann surface of large genus" by Buser, P. and Sarnak, P., pp 44-45

<u>Goal:</u> find surfaces S_{g_k} with systole $\gtrsim \frac{4}{3}\log(g_k)$ with genus g_k.

<u>Recall:</u> $\mathbb{H}^{a,b}_{\mathbb{Q}} = \{x_0 + ix_1 + jx_2 + kx_3 \mid x_i \in \mathbb{Q}\}$ $\quad i^2 = a,\ j^2 = b,\ ij = -ji = k$

$\mathbb{Q}$ is in the center of $\mathbb{H}^{a,b}_{\mathbb{Q}}$

(*) $N_{red}(x) = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2$. If this is $\neq 0$ for any nontrivi $\Rightarrow \mathbb{H}^{a,b}_{\mathbb{Q}}$ div. alg.

<u>Def.</u> $\underline{\tilde{\Gamma}} := \underline{SL(1, \mathbb{H}^{a,b}_{\mathbb{Z}})} := \{g \in \mathbb{H}^{a,b}_{\mathbb{Z}} \mid N_{red}(g) = 1\}$

This is cocpt if a, b are as in (*).

<u>Def.</u> p odd prime

$$\tilde{\Gamma}(p) := \{g \in SL(1, \mathbb{H}^{a,b}_{\mathbb{Z}}) \mid \underbrace{g \equiv 1 \pmod p}_{\substack{x_0 \equiv 1 \pmod p,\\ x_1 \equiv x_2 \equiv x_3 \equiv 0 \pmod p}}\}$$

p^{th} <u>principal congruence subgp of Γ</u>

for $g = x_0 + x_1 i + x_2 j + x_3 k$

$$\tilde{\Gamma}, \tilde{\Gamma}(p) \hookrightarrow SL(2,\mathbb{R}) \twoheadrightarrow PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/_{\pm Id}$$

$$x_0 + x_1 i + x_2 j + x_3 k \longmapsto \begin{pmatrix} x_0 + x_1\sqrt{a} & x_2 + x_3\sqrt{a} \\ b(x_2 - x_3\sqrt{a}) & x_0 - x_1\sqrt{a}\end{pmatrix}$$

<u>Lemma 1.</u> $\tilde{\Gamma}(p)$ is torsion free, i.e. $\nexists\, \gamma \in \tilde{\Gamma}(p)\setminus\{1\},\ n \in \mathbb{N}_{>0}: \gamma^n = 1$.

Pf: <u>Case 1:</u> $p \nmid n$, $\gamma \in \tilde{\Gamma}(p)\setminus\{1\}$

$\Rightarrow \gamma = 1 + y_0 p^{d_0} + y_1 p^{d_1} i + y_2 p^{d_2} j + y_3 p^{d_3} k$, $\quad p \nmid y_0, y_1, y_2, y_3$,

not all y_r are 0 and $\forall d_r > 0$.

We will show $\gamma^n \not\equiv 1 \pmod{p^{d+1}}$ where $d = \min\{d_r \mid y_r \neq 0,\ r=0,1,2,3\} \geq 1$

$$\gamma^n = 1 + n(y_0 p^{d_0} + y_1 p^{d_1} i + y_2 p^{d_2} j + y_3 p^{d_3} k) + \text{higher order terms, all divisible by } p^{d+1}$$

$\not\equiv 1 \pmod{p^{d+1}}$ because $p \nmid n$.

Case 2. $p \mid n$.

Write $\gamma^n = \gamma^{p^s n'}$, $p \nmid n'$.

By Case 1: $\gamma^{p^s} \neq 1$. $\quad \gamma^{p^s} = \underbrace{((\gamma^p)^p \ldots)^p}_{s \text{ times}}$

Hence it suffices to show $\gamma^p \neq 1$.

$p^{d+1} \mid \binom{p}{i} p^{di} \quad \forall i > 1$, binomial term.

$\Rightarrow \gamma^p = 1 + p(y_0 p^{d_0} + \ldots + y_3 p^{d_3} k)$
$\not\equiv 1 \mod p^{d+2}$

Cor. $\Gamma(p) \backslash \mathbb{H}^2$ is a closed hypersurface.

Lemma 2. $\tilde{\Gamma} / \tilde{\Gamma}(p) = SL(2, \mathbb{Z}/p\mathbb{Z})$

Pf: We have a map $\mathbb{H}^{a,b}_{\mathbb{Z}} \twoheadrightarrow \mathbb{H}^{a,b}_{\mathbb{Z}/p\mathbb{Z}}$ (reducing mod p)

which restricts to a map $SL(1, \mathbb{H}^{a,b}_{\mathbb{Z}}) \longrightarrow \mathbb{H}^{a,b}_{\mathbb{Z}/p\mathbb{Z}}$

The img of the latter is $SL(2, \mathbb{Z}/p\mathbb{Z})$ and the kernel is $\tilde{\Gamma}(p)$
(using $a, b \not\equiv 0\ (p)$)

Lemma 3. $\# SL(2, \mathbb{Z}/p\mathbb{Z}) = p(p+1)(p-1)$

Pf: Start with $\# GL(2, \mathbb{Z}/p\mathbb{Z})$

$$GL(2, \mathbb{Z}/p\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z}/p\mathbb{Z},\ ad - bc \neq 0 \right\}$$

If $a = 0$ or $d = 0 \longrightarrow b, c \neq 0$

$\underbrace{(2p-1)}_{a \text{ and } d} \cdot \underbrace{(p-1)^2}_{b \text{ and } c \text{ can be chosen freely}}$

If $ad \neq 0 \longrightarrow \underbrace{(p-1)^2}_{a,d \text{ chosen freely}} \cdot (\underset{\text{if } b=0}{p} + \underbrace{(p-1)^2}_{\text{if } b \neq 0 \text{ then } c \neq \frac{ad}{b}})$

$\Big\}\ \# GL(2, \mathbb{Z}/p\mathbb{Z}) = (p-1)^2 p(p+1)$

$$\det: GL(2, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathbb{Z}/p\mathbb{Z} \setminus \{0\}, \qquad SL(2, \mathbb{Z}/p\mathbb{Z}) = \ker(\det)$$

$$\Rightarrow \# SL(2, \mathbb{Z}/p\mathbb{Z}) = \frac{\# GL(2, \mathbb{Z}/p\mathbb{Z})}{\#\left((\mathbb{Z}/p\mathbb{Z})^\times\right)} = (p-1)\,p\,(p+1)$$

□

<u>Lemma 4.</u> $\exists A > 0$ s.t. $\forall p >$ a.b prime:

$$g_p = \text{genus}\left(\Gamma(p)\backslash \mathbb{H}^2\right) = p(p-1)(p+1)A + 1$$

Pf: Set $A' = \text{area}\left(\Gamma \backslash \mathbb{H}^2\right)$ (There is a formula for this by Shimizu and Borel.)

$$\text{area}\left(\Gamma(p)\backslash \mathbb{H}^2\right) \underset{\substack{\uparrow \\ \text{covering space}}}{=} [\Gamma : \Gamma(p)] \cdot \underbrace{\text{area}\left(\Gamma\backslash \mathbb{H}^2\right)}_{A'} = p(p+1)(p-1)A' \cdot \frac{1}{2}$$

$\Gamma/\Gamma(p) \simeq PSL(2, \mathbb{Z}/p\mathbb{Z})$

(This is not quite precise since $\Gamma\backslash \mathbb{H}^2$ may not be a mfold.)

$$\text{area}\left(\Gamma(p)\backslash \mathbb{H}^2\right) = \int_{\Gamma(p)\backslash \mathbb{H}^2} K = 2\pi\,\chi\left(\Gamma(p)\backslash \mathbb{H}^2\right) = 4\pi(1-g_k) \quad \text{by Gauss-Bonnet}$$

<u>Lemma 5.</u> $\text{sys}\left(\Gamma(p)\backslash \mathbb{H}^2\right) \geq 2\cosh^{-1}(p^2-1)$

Pf: Recall $\ell(\gamma) = 2\cosh^{-1}\left(\frac{|\operatorname{tr}\gamma|}{2}\right)$ for a closed geodesic γ.

Suppose $x_0 + x_1 i + x_2 j + x_3 k \in \tilde{\Gamma}(p) \Rightarrow p \mid x_1, x_2, x_3$

and $1 = x_0^2 - a x_1^2 - b x_2^2 + ab\, x_3^2 \Rightarrow x_0^2 \equiv 1 \mod p^2$

$\Rightarrow x_0 \equiv \pm 1 \mod p^2$ since $(\pm 1 + mp)^2 = 1 \pm 2mp + m^2p^2$

$$|\operatorname{tr}(g)| = \left|\operatorname{tr}\begin{pmatrix} x_0 + x_1\sqrt{a} & x_2 + x_3\sqrt{a} \\ b(x_2 - x_3\sqrt{a}) & x_0 - x_1\sqrt{a}\end{pmatrix}\right| = |2x_0| \geq 2p^2 - 2$$

by $\Gamma(p)\backslash \mathbb{H}^2$ being cpt and the following exc.

Exc. If A is a 2×2 matrix with $\det A = 1$, $\operatorname{tr} A = 2 \Rightarrow A$ is unipotent □

<u>Cor.</u> $\exists C > 0$ constant s.t. $\text{sys}\left(\Gamma(p)\backslash \mathbb{H}^2\right) \geq \frac{4}{3}\log(g_p) - C \quad \forall p >$ a.b prime

Pf: $g_p \leq A'' p^3 \quad \Rightarrow \text{sys} \geq 2\log p^2$

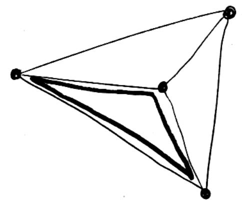
$$\Rightarrow \text{sys}\left(\Gamma(p)\backslash \mathbb{H}^2\right) \geq 2\log\left(g_p^{1/2} A'''\right) = \frac{4}{3}\log(g_p) - 2\log(A''')$$

□

<u>Thm.</u> (Mumford) $\sup\{\text{sys}(S) \mid S \text{ closed hyp. surface of genus } g\}$ is realised.
PO

Def $\mathcal{G}$ graph is called k-regular if $\deg(v) = k$ $\forall v$ vertex of $\mathcal{G}$.

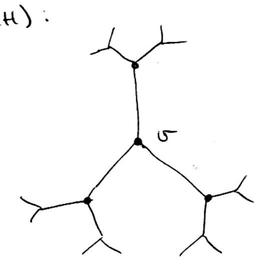
Def. girth $(\mathcal{G}) :=$ shortest cycle in $\mathcal{G}$



Lemma. $\mathcal{G}$ a k-regular graph on n vertices.

$$\Rightarrow \quad \text{girth}(\mathcal{G}) \lesssim 2 \log_{k-1}(n).$$

PF (SKETCH):



Take the "disc" around the vertex, count the vertices, they cannot coincide.

Thm. (Lubotzky – Phillips – Sarnak) p odd prime $\Rightarrow \exists (p+1)$-regular graph $\mathcal{G}$ with girth$(\mathcal{G}) \gtrsim \frac{4}{3} \log_p (\#\text{vert } \mathcal{G})$

21.06.2018

Lattices in $SO(1,n)$

Recall that $\left(SO(1,n) / \pm Id \right)^0 = PO(1,n)^0 = \text{Isom}^+(\mathbb{H}^n)$

where $\mathbb{H}^n = \{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 - x_1^2 - \dots - x_n^2 = 1, \ x_0 > 0 \}$

Prop. Let $a_1, \dots, a_n \in \mathbb{Z}_{>0}$, $G = SO(x_0^2 - a_1 x_1^2 - \dots - a_n x_n^2 ; \mathbb{R}) \simeq SO(1,n)$.
If $n \geq 4$ then $G_{\mathbb{Z}}$ is an arithmetic lattice that is not compact.

PF (SKETCH): G def over $\mathbb{Q}$ $\Rightarrow$ $G_{\mathbb{Z}}$ is an arith subgp.

Meyer's thm.: $\forall n \geq 4$, $a_i \in \mathbb{Q}$: $x_0^2 - a_1 x_1^2 - \dots - a_n x_n^2 = 0$ has a nontrivial integral solution.

The same argument implies the compactness of $G_{\mathbb{Z}}$.

Prop: If $n \notin \{3, 7\}$ then up to conjugation and commensurability these are all the non-compact arithmetic groups in $SO(1,n)$.

Prop. Let $F \neq \mathbb{Q}$ be a totally real algebraic number field, $\mathcal{O}_F$ the ring of integers, $a_1, \dots, a_n \in F^+$ s.t. $\forall \sigma \neq id$ place $\sigma(a_i) < 0$ $\forall i = 1, \dots, n$,
$G = SO(x_0^2 - a_1 x_1^2 - \dots - a_n x_n^2, \mathbb{R}) \simeq SO(1,n)$ $\Rightarrow$ $G_{\mathcal{O}_F}$ is a cocpt ar. subgp of G.

PF: Same as in 2-dim case.

Prop. For $2 | n$ these are up to conjugacy and commensurability all the cocpt arith subgps of $SO(1,n)$.

For $2 \nmid n$ we need a second construction using quaternion algebras.

Question. Are there any non-arithmetic lattices?

Yes $\longrightarrow$ Gromov – Radetski – Shapiro

Construction of non-arithmetic lattices $n \geq 3$

Def. A connected Riemannian mf. M is called hyperbolic if

1) its universal cover $\tilde{M}$ is isometric to $\mathbb{H}^n$

2) M is orientable

3) M is compact (is this implied by 1)?)

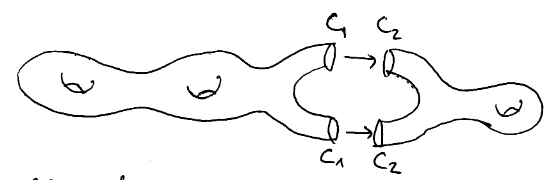
Prop. A connected Riemannian mf. M of finite volume is hyperbolic iff $\exists \Gamma < SO(1,n)^\circ$ torsion-free lattice lattice s.t. M is isometric to $\Gamma \backslash \mathbb{H}^n$.

Idea. Build hybrid manifolds: not arithmetic but built up from arithmetic pieces.

Hybrid manifolds

C_1, C_2 totally geodesic, $\exists$ isometry $C_1 \longrightarrow C_2$

Γ_1, Γ_2 not commensurable

Cut and glue: 

$\rightsquigarrow M$

This works b/c we have codim 1.

Prop. Suppose
- M_1, M_2 are connected mfs w/ bdry,
- $f: \partial M_1 \longrightarrow \partial M_2$ a homeo.

Then define $M_1 \cup_f M_2$; this is an n-dim conn. mf. w/o bdry. $\square$

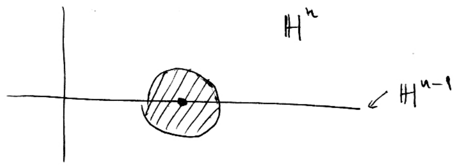
This means that topologically our construction works; the problem is if it works geometrically.

Problem. Gluing 2 Riemannian mfs along bdry does not always give a Riemannian mf., even if the gluing map is an isometry.

Ex. $D_1, D_2 \underset{\text{isom}}{\cong} \{ z \in \mathbb{R}^2 \mid \|z\| = 1 \}$, $f: \partial D_1 \longrightarrow \partial D_2$

If we could extend the metric to $D_1 \underset{f}{\cup} D_2$ then we would get a flat metric on S^2 which cannot occur by Gauss–Bonnet.

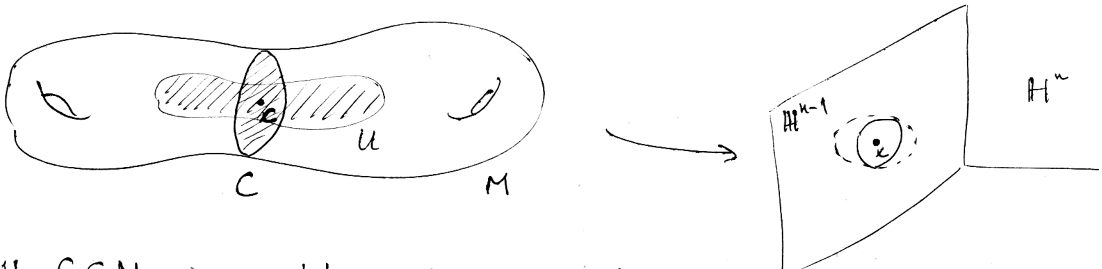
So when is this possible? Intuitively things have to look like:



This means the following formally:

Def. M a hyperbolic mf. A totally geodesic hypersurface in M is a closed non-empty connected submf. $C \subseteq M$ s.t. $\forall c \in C$:

- $\exists U \subseteq M$ open nbh. of c
- $\exists x \in \mathbb{H}^{n-1} \subseteq \mathbb{H}^n$
- $\exists V \subseteq \mathbb{H}^n$ nbhd of x
- $\exists g: U \longrightarrow V$ Riem isometry s.t. $g(c) = x$, $g(U \cap C) = \mathbb{H}^{n-1} \cap V$.



Rmk. If $C \subseteq M$ is a tot. geod. hypersurface in a hyp. n-mf. of finite volume, then there are

- $\Gamma < PO(n,1)$ lattice
- $f: M \longrightarrow {}_\Gamma\backslash^{\mathbb{H}^n}$ isometry s.t. $f(C)$ is the img of $\mathbb{H}^{n-1}$ in $\Gamma\backslash\mathbb{H}^n$

In particular, C is a hyperbolic (n−1)-mf.

Prop. If
- M_1, M_2 are hyp. n-manifolds of finite volume,
- $C_j \subseteq M_j$ totally geodesic hypersurfaces s.t. $M_j \setminus C_j$ is connected
- $f: C_1 \longrightarrow C_2$ a Riemannian isometry

then $\overline{M_1 \setminus C_1} \cup_{f,f} \overline{M_2 \setminus C_2}$ is a hyp. mnf. where $\overline{M_j \setminus C_j}$ is $M_j \setminus C_j$ without two copies of C_j attached as boundary and $\cup_{f,f}$ means that we use two copies of f for gluing.

Pf: Picture + EXC. 6.5.4. $\square$

How do we obtain totally geodesic hypersurfaces?

Lemma: Suppose we have
- $\Gamma < PO(1,n)^0$ torsion free lattice
- C the img of $\mathbb{H}^{n-1}$ in $\Gamma \backslash \mathbb{H}^n$
- $\tau: \mathbb{H}^n \longrightarrow \mathbb{H}^n$ reflection across $\mathbb{H}^{n-1}$, i.e. $\tau(x_0, \dots, x_{n-1}, x_n) = (x_0, \dots, x_{n-1}, -x_n)$

(conversely, the isometry of $\mathbb{H}^{n-1}$ can be extended to perserve isometry of $\mathbb{H}^n$ (?))

such that the following hold:
- $\Gamma \cap PO(1, n-1)$ is a lattice
- $\exists \Gamma' < PO(1,n)^0$ torsion-free lattice s.t.
  - Γ' is normalised by τ, i.e. $\tau \Gamma' \tau = \Gamma'$
  - $\Gamma < \Gamma'$

Then C is a totally geodesic hypersurface.

Pf: From the setup we get that if C is a closed embedded submanifold then it is a totally geodesic hypersurface. Thus it sts C is a closed embedded submf.

$\Gamma_0 := \{ \gamma \in \Gamma \mid \gamma \mathbb{H}^{n-1} = \mathbb{H}^{n-1} \} \Rightarrow \Gamma \cap PO(1, n-1) < \Gamma_0$ is of index ≤ 2

$\Rightarrow \Gamma_0$ is a lattice.

By EXC. 4.4.3 we get a map $\varphi: \Gamma_0 \backslash \mathbb{H}^n \longrightarrow \Gamma \backslash \mathbb{H}^n$ a proper embedding.

(Proper: preimg of cpt is cpt. If $f: X \to Y$ is proper, Y loc. cpt. T2 $\Rightarrow$ f closed.)

We need to show injectivity.

Suppose $\gamma x = y$ for some $\gamma \in \Gamma$, $x, y \in \mathbb{H}^{n-1}$

WTS: $\gamma \in \Gamma \cap PO(n,1)$

We have $\gamma^{-1}\tau\gamma\tau \in \Gamma'$ and $\gamma^{-1}\tau\gamma\tau(x) = x$

Γ' is torsion-free $\Rightarrow$ no elt of $\Gamma' \setminus \{e\}$ has a fixed pt. in $\mathbb{H}^n$

$\Rightarrow \gamma^{-1}\tau\gamma\tau = e \Rightarrow \gamma$ commutes with $\tau \overset{\text{EXC.}}{\Longrightarrow} \gamma$ preserves the fixed points of $\tau \Rightarrow \gamma$ preserves $\mathbb{H}^{n-1} \rightarrow \gamma \in \Gamma \cap PO(1, n-1)$ □

26.06.2018

Thm. Suppose we have

- M_1, M_2 hyperbolic mfs of finite volume
- $C_j \subseteq M_j$ totally geodesic Riemannian hypersurface
- $f: C_1 \longrightarrow C_2$ Riem. isometry
- C_1, C_2 have finite volume as hyperbolic $(n-1)$-mfs.
- Both $M_1 \setminus C_1$ and $M_2 \setminus C_2$ are connected.

Then if $\overline{M_1 \setminus C_1} \cup_{f, f} \overline{M_2 \setminus C_2}$ is arithmetic then M_1 is commensurable to M_2.

Cor. $\forall n \geq 3$ odd there exist non-cocpt non-arithmetic lattices in $PO(1,n)$

(The assumptions $2 \nmid n$ and non-cocptness are unnecessary, see the EXERCISES in § 6.5.)

Prop. $B_1, B_2 : \mathbb{Q}^{n+1} \longrightarrow \mathbb{Q}$, $\quad B_1(x) := x_0^2 - x_1^2 - \dots - x_n^2$

$$B_2(x) := x_0^2 - x_1^2 - \dots - x_{n-1}^2 - 2x_n^2$$

Let $\Gamma_1 := SO(B_1, \mathbb{Z})$, $\Gamma_2 := SO(B_2, \mathbb{Z})h^{-1}$ where $h := \operatorname{diag}(1, \dots, 1, \sqrt{2})$

(Thus both $\Gamma_1, \Gamma_2 < SO(1,n)$.) Let $M_j := \Gamma_j \backslash \mathbb{H}^n$, $C_j :=$ img of $\mathbb{H}^{n-1}$ in M_j,

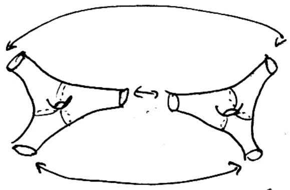
$\hat{\Gamma}_j := \Gamma_j \cap SO(1, n-1)$.

Issues:
- are $M_j \setminus C_j$ connected?
- are $\Gamma_1 \approx \Gamma_2$ commensurable?

Step 1. Pass to finite index subgps s.t. Γ_j are torsion free

Both $SO(B_j, \mathbb{Z})$ are normalised by τ, hence C_j tot. geod. by lemma

Claim. By passing to finite index subgroups wma $M_j \setminus C_j$ to be connected.



Goal. Imitate this approach

Find finite cover $\tilde{M}_j \xrightarrow{\pi} M_j$

s.t. $\pi^{-1}(C_j)$ has multiple conn. components.

PF OF CLAIM: Let $R_j := \mathbb{Z}[\text{matrix entries of } \Gamma_j] \subseteq \mathbb{R}$

This is finitely generated because Γ_j is (this follows from a thm. we didn't do in class.) R_j is an integral domain with 1

$\forall x \in R_j \setminus \{0\}$ $\exists I_j \subseteq R_j$ $\overset{\text{max.}}{\text{ideal}}$ s.t. $x \notin I_j$ (This is true because $\mathbb{Z}$ is a Jacobson ring, every finitely generated ring over a Jacobson ring is Jacobson itself, so in particular, R_j is Jacobson, i.e. the intersection of all maximal ideals is $\{0\}$. Note that finitely generatedness is essential: $0 \neq p \in \mathbb{Z}_p$ is contained in the unique max. ideal $p\mathbb{Z}_p$.)

Let $g \in \Gamma_j \setminus \hat{\Gamma}_j$ $\Rightarrow$ $\exists$ matrix entry g_{kl} s.t. $\tau(g)_{kl} \neq g_{kl}$

$\Rightarrow$ $x := \tau(g)_{kl} - g_{kl} \in R_j \setminus \{0\}$. Let I_j be a max. ideal s.t. $x \notin I_j$.

Consider the map

$$\Gamma_j < SO(1,n)_{R_j} \longrightarrow SO(1,n)_{R_j/I_j} \times SO(1,n)_{R_j/I_j}$$

$$\gamma \longmapsto ([\gamma]_{I_j}, [\tau(\gamma)]_{I_j})$$

Note that $\hat{\Gamma}_j$ lands in the diagonal while Γ_j does not (by construction).

We obtain a finite quotient of Γ_j in which $\hat{\Gamma}_j$ maps to a smaller finite group. $\Rightarrow$ The finite cover of M_j corresponding to this quotient has multiple copies of C_j $\Rightarrow$ one of them will no longer separate $\boxed{C}$

Thus we have $\Gamma_1 \approx \Gamma_2$. Exc. 6.5.9: we can pass to further finite cover to make C_1 and C_2 isometric.

We can perform gluing: get $M = \overline{M_1 \setminus C_1} \underset{\text{fit}}{\cup} \overline{M_2 \setminus C_2}$ (noncompact)

Goal: show that M is non-arithmetic, i.e. show $M_1 \underset{\text{comm.}}{\not\sim} M_2$.

Step 1. We can recover the quadratic form from Γ_1.

Pf. Exc. 6.4.2: $G < GL(\ell, \mathbb{C})$ Lie group that does not fix a proper nonzero subspace of $\mathbb{C}^\ell$ and B_1, B_2 are G-invariant quadratic forms then $\exists \lambda \in \mathbb{C}$ s.t. $B_1 = \lambda B_2$.

Γ lattice + Borel density $\Rightarrow$ quad. form fixed by Γ is unique up to multiples.

Suppose $\exists g \in O(1,n)$ s.t. $g h \Gamma_2 h^{-1} g^{-1}$ commensurable to Γ_1

$\Rightarrow$ $g' = gh$ has $(g')^t \operatorname{diag}(-1, 1, \dots, 1, 2)\, g' = \lambda \operatorname{diag}(-1, 1, \dots, 1)$ where $\lambda \in \mathbb{Q} \setminus \{0\}$.

Consider the discriminant of B_j $\det(B_j)$

$\operatorname{disc}(B_1) = -1$, $\operatorname{disc}(B_2) = -2$

$\lightning$ Commensurability $\Rightarrow$ $-e\lambda^{n+1} = -1$ $\Rightarrow$ 2 is a square in $\mathbb{Q}$ $\lightning$

Hence M is non-arithmetic.

PF OF COMMENSURABILITY THM:

Lemma. If
- G has no cpt factors
- Γ_1, Γ_2 arithmetic lattices in G
- $\Gamma_1 \cap \Gamma_2$ Zariski dense in G

then Γ_1, Γ_2 are commensurable.

PF OF LEMMA: By def we have isomorphisms $\varphi_j : G \xrightarrow{\sim} H_j$ s.t. $\varphi_j(\Gamma_j) = (H_j)_{\mathbb{Z}}$

Define $\varphi : G \longrightarrow H_1 \times H_2$ i.e. $\varphi = \varphi_1 \times \varphi_2$.

$$g \longmapsto (\varphi_1(g), \varphi_2(g))$$

$\varphi(G)_{\mathbb{Z}} = \varphi(\Gamma_1 \cap \Gamma_2)$ Zariski dense in $\varphi(G)$

Prop. 5.1.8 $\Rightarrow$ $\varphi(G)$ def'd over $\mathbb{Q}$ $\Rightarrow$ $\varphi(G)_{\mathbb{Z}}$ is a lattice

$\Rightarrow \Gamma_1 \cap \Gamma_2 < \Gamma_1$ is of finite index
$\qquad\qquad < \Gamma_2$

Thm. (mild generalisation of what was stated before)

- M_1, M_2 n-manifolds
- $M_j' \subset M_j$ submf. with non-empty totally geodesic boundary
- $f: \partial M_1' \longrightarrow \partial M_2'$ isometry
- $\partial M_j'$ has finitely many conn. components, all of which have finite volume

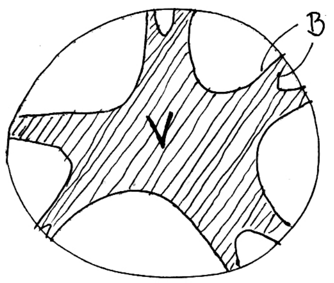
If $M = M_1' \underset{f}{\cup} M_2'$ is arithmetic then $M \approx M_1$. Change "manifold" to "arithmetic hyperbolic manifold"!

Pf: Setup.

$M =: \Gamma \backslash \mathbb{H}^n$, $\varphi: \mathbb{H}^n \longrightarrow M$ covering map

$B := \varphi^{-1}(\partial M_1') \subseteq \mathbb{H}^n$

$V :=$ closure of a component of $\mathbb{H}^n \setminus B$ containing a point from $\varphi^{-1}(M_1')$



dim = 2

$$\Gamma' := \{\gamma \in \Gamma \mid \gamma V = V\}$$

Exc. 6.5.12

$$= \{\gamma \in \Gamma \mid \operatorname{int}(\gamma V \cap V) \neq \emptyset\}$$

Since $M_1' = \varphi(V)$ we get $M_1' = \Gamma' \backslash V$

Def. $U \subseteq \mathbb{H}^n$ is <u>convex</u> if $\forall x, y \in U$ the geodesic segment between x and y is contained in U.

V is the intersection of half-spaces $\Rightarrow$ V is convex $\Rightarrow$ V is simply connected $\Rightarrow$ V is the universal cover of M_1'.

Similarly, write $M_1 = \Gamma_1 \backslash \mathbb{H}^n$, φ_1 the covering map, B_1 the lift of the boundary of M_1, V_1 the univ cover of M_1, Γ_1' the corresponding subgp.

By the uniqueness of the universal cover there is a Riemannian isometry $V \longrightarrow V_1$.

We can move every totally geodesic copy of $\mathbb{H}^{n-1}$ in $\mathbb{H}^n$ to "the standard" copy with an isometry of $\mathbb{H}^n$, where "the standard" copy is

$$\mathbb{H}^{n-1} = \{(x_0, \ldots, x_n) \in \mathbb{R}^n \mid x_0 > 0,\ x_n = 0,\ x_0^2 - x_1^2 - \ldots - x_n^2 = -1\}$$

Thus we may set everything up in such a way that the standard $\mathbb{H}^{n-1}$ is one of the ∂-components of V and the isometry $V_1 \longrightarrow V$ is the identity.

$\Rightarrow \Gamma_1' = \Gamma' = \Gamma \cap \Gamma_1$

Recall the following:

Lemma. If • G has no cpt factors,

• Γ_1, Γ_2 are arith. lattices in G

• $\Gamma_1 \cap \Gamma_2$ is Zariski dense in Γ_1

then Γ_1 and Γ_2 are commensurable.

Goal: the Zariski closure of Γ' contains $PO(n,1)^\circ$.

Then it follows from the lemma that $\Gamma_1 \approx \Gamma$ and hence $M_1 \approx M$.

Pf: $\partial M_1'$ fin. vol. $\Rightarrow$ $\Gamma' \cap SO(1,n-1)$ is a lattice

Borel density thm. $\Rightarrow$ $\bar{\Gamma}'$ is the Zariski closure of Γ', $\bar{\Gamma}'^\circ \supseteq PO(1,n)^\circ$

Fact. (Exc. 6.5.13) $PO(1,n-1)^\circ$ is a maximal connected subgroup of $PO(1,n)^\circ$

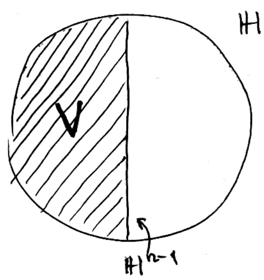
So if $\bar{\Gamma}'^\circ \supsetneq PO(1,n-1)^\circ$ then $\bar{\Gamma}'^\circ = PO(1,n)^\circ$

Suppose $\bar{\Gamma}'^\circ = PO(1,n-1)^\circ$. Since $\bar{\Gamma}'^\circ < \bar{\Gamma}'$ has finite index, $PO(1,n-1)^\circ$ contains a finite index subgp of $\bar{\Gamma}'$.

In fact, $\{\gamma \in \Gamma' \mid \gamma H = H\}$ has finite index in Γ' $\forall H \subseteq \partial V$ connected component.

Claim. This leads to a contradiction.

Case 1. ∂V is connected



Up to finite index: $\Gamma' < PO(1,n-1)$

Let $\tau : \mathbb{H}^n \longrightarrow \mathbb{H}^n$

$(x_0,\dots,x_n) \longmapsto (x_0,\dots,x_{n-1},-x_n)$

τ centralises Γ'

$\mathbb{H}^n = V \cup \tau(V)$

$\Gamma' \backslash V$ and $\Gamma' \backslash \tau(V)$ are fin vol hyp mfs. $\Rightarrow$ $\Gamma' \backslash \mathbb{H}^n$ is a fin vol hyp mf.

$\Rightarrow \Gamma'$ is a lattice in $PO(1,n)$, contradicting Borel density $\lightning$

Case 2a. ∂V not connected, $\partial M_1'$ cpt.

H_1, H_2 two distinct conn. components of ∂V

$\Gamma' \backslash H_1$ is compact $\Rightarrow$ $\exists$ compact subset $C \subseteq H_1$ s.t. $\Gamma' C = H_1$

$\delta := \min \{ \mathrm{dist}(c, H_2) \mid c \in C \} > 0$

Since Γ acts by isometries, $\delta = \mathrm{dist}(H_1, H_2)$.

Fact. (negative curvature) $\exists ! \, p \in H_1$ s.t. $\mathrm{dist}(p, H_2) = \delta$.

$\Rightarrow \forall \gamma \in \Gamma'$ fixes p, and since Γ acts by isometries, $\Gamma = \{e\}$ $\lightning$
and Γ' acts freely

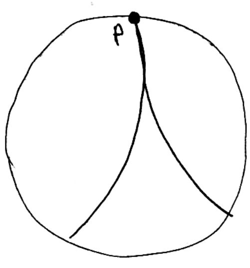
Something is fishy, infinite indices may be at play.

When $H_2 = \mathbb{H}^n$, and since $PO(1, n-1)^\circ$ contains a finite index subgp of Γ' things will work out.

Case 2b ∂V not connected, $\partial M_1'$ non-cpt.

The proof is similar: if H_1, H_2 have positive distance then the argument from 2a works.

When no such pair of conn. components exist.



Hyperbolic geometry $\Rightarrow$ they must intersect at a single point $p \in \partial_\infty \mathbb{H}^n$

This is a fixed point.

$\xrightarrow{\text{Exc.}}$ Γ' is not Zariski dense. $\lightning$ This finishes the pf of the thm. $\square$

Noncompact arithmetic subgroups of $SL(3, \mathbb{R})$

Prop. Let

- $L := \mathbb{Q}(\sqrt{r})$ for $r \geq 2$ square-free integer
- $\sigma \neq \mathrm{id}$ place of L
- $J_3 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
- $\Gamma = SU(J_3, \sigma, \mathbb{Z}[\sqrt{r}]) = \{ g \in SL(3, \mathbb{Z}[\sqrt{r}]) \mid \sigma(g^T) J_3 \, g = J_3 \}$

Then 1) Γ is an arith subgp of $SL(3, \mathbb{R})$

2) Γ is not cocpt

3) Γ is not commensurable to $SL(3, \mathbb{Z})$.

Pf: $\Delta : L^3 \longrightarrow \mathbb{R}^6$, $\Delta(v) = (v, \mathcal{F}_3(v))$

$V_{\mathbb{Q}} := \Delta(L^3)$

$\mathcal{L} := \mathbb{Z}[\sqrt{r}]^3$

$\rho : SL(3,\mathbb{R}) \longrightarrow SL(6,\mathbb{R})$ defined by $\rho(A)(v,w) = (Av, (A^T)^{-1} w)$ $\forall v, w \in \mathbb{R}^3$

Then
- $V_{\mathbb{Q}}$ is a $\mathbb{Q}$-form for $\mathbb{R}^6$
- $\mathcal{L}$ is a $\mathbb{Z}$-lattice for $V_{\mathbb{Q}}$
- ρ is a homomorphism
- $\Gamma = \{ g \in SL(3,\mathbb{R}) \mid \rho(g)\mathcal{L} = \mathcal{L} \}$

To prove 1) we nts that $\rho(SL(3,\mathbb{R}))$ is def. over $\mathbb{Q}$.

Recall: nts $\rho(SL(3,\mathbb{R}))_{\mathbb{Q}}$ is dense in $\rho(SL(3,\mathbb{R}))$.

Let $U_1 := \left\{ \begin{pmatrix} 1 & a+b\sqrt{r} & \frac{-a^2-rb^2}{2} + c\sqrt{r} \\ 0 & 1 & -a+b\sqrt{r} \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a,b,c \in \mathbb{Q} \right\}$ i.e. we work with unipotents in extreme explicity

$\rho(U_1) \subseteq \rho(SL(3,\mathbb{R}))_{\mathbb{Q}}$ and U_1 is dense in $U := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$

Likewise we may construct $U_2 \subseteq U^T = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} \right\}$ dense.

$\langle U, U^T \rangle = SL(3,\mathbb{R})$ ✓

2): elts in U_Γ are contained in $\rho(\Gamma)$ and unipotent.

Godement $\Rightarrow$ non-cocpt. ✓

3): Sketch: • U_Γ maximal unipotent subgroup of Γ.

Choose $w \in \mathbb{Z}[\sqrt{r}] \setminus \{0\}$ s.t. $\sigma(w) = \frac{1}{w}$

$h := \operatorname{diag}(w, 1, w^{-1})$ satisfies $h U_\Gamma h^{-1} = U_\Gamma$ (just computation)

$\Rightarrow U_\Gamma$ has infinite index in $N_\Gamma(U_\Gamma)$.

• $U_{\mathbb{Z}}$ max. unipotent subgp of $SL(3,\mathbb{Z})$

If $U'_{\mathbb{Z}} < SL(3,\mathbb{Z})$ commensurable to $U_{\mathbb{Z}}$ then $[N_{SL(3,\mathbb{Z})}(U'_{\mathbb{Z}}), U'_{\mathbb{Z}}] < \infty$

and all maximal unipotent subgroups of $SL(3,\mathbb{Z})$ are conjugate.

Ch. 7. $SL(n,\mathbb{Z}) < SL(n,\mathbb{R})$ is a lattice (finally!)

03.07.2018

Three main ingredients:

(a) Siegel sets (definition)

(b) Prove that they have finite measure

(c) Siegel sets are coarse fundamental domains for $SL(n,\mathbb{Z}) \curvearrowright SL(n,\mathbb{R})$, i.e. larger than an actual fund. domain but still sufficient for our purposes

This is called reduction theory. The goal is to make every elt in $SL(n,\mathbb{R})$ as simple as possible using $SL(n,\mathbb{Z})$.

§7.1. Iwasawa decomposition

Thm. (Iwasawa decomposition) $G = SL(n,\mathbb{R})$. Then $G = KAN$ where

$$K = SO(n),\quad A = \left\{\begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn}\end{pmatrix}\right\}^0,\quad N = \left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1\end{pmatrix}\right\}.$$

Moreover, $\forall g \in G$ has a unique representation $g = kan$ with $k \in K, a \in A, n \in N$.

Pf.: Uniqueness: $k_1 a_1 n_1 = k_2 a_2 n_2 \Rightarrow k_1^{-1} k_2 = a_1 n_1 n_2^{-1} a_1^{-1} \in K \cap AN$

This follows from $AN = NA$ which is easily seen from def.

Claim: $K \cap AN = \{e\}$.

$x \in K \cap AN \Rightarrow$ all eigenvalues are ± 1 as it is in SO

But eigenvalues of an upper diag matrix are the elts of the diagonal, hence all eigenvalues are $+1$. Since $x \in SO(n)$, this implies $x = e$. ✓

$\Rightarrow k_1 = k_2 \Rightarrow a_1^{-1} a_2 = n_1^{-1} n_2 \in A \cap N = \{e\}$ ✓

Existence. Take $e_1, \dots, e_n$ to be the std basis of $\mathbb{R}^n$.

$\forall g \in G$: $\{g e_i\}_i$ is a basis of $\mathbb{R}^n$. Let $v_i := g e_i$

Gram–Schmidt: $w_i^* = v_i - \sum_{j=1}^{i-1} (v_i, w_j)\, w_j$, $\quad w_i := \frac{1}{\|w_i^*\|} \cdot w_i^*$

$\{w_i\}_i$ ONB.

$\Rightarrow \exists k \in O(n)$ s.t. $k^{-1} e_i = w_i \quad \forall i = 1, \dots, n$

$\Rightarrow \exists a \in A$ s.t. $k w_i^* = a e_i$ (note that the norm came back)

By an inductive argument we get that $w_i \in \operatorname{span}_{\mathbb{R}}\{v_1, \dots, v_i\}$.

$\Rightarrow g^{-1} w_i^* = g^{-1} v_i - g^{-1} \sum_{j=1}^{i-1} (v_i, w_i)\, w_j \in g^{-1} v_i - g^{-1} \operatorname{Span}_{\mathbb{R}}\{v_1, \dots, v_{i-1}\}$

reversing the GS step $\qquad = \varepsilon_i - \operatorname{span}_{\mathbb{R}}\{\varepsilon_1, \dots, \varepsilon_{i-1}\}$

$\Rightarrow \exists u \in N: \quad g^{-1} w_i^* = u\varepsilon_i \qquad \forall i = 1, \dots, n$

$\Rightarrow u^{-1} g^{-1} w_i^* = \varepsilon_i = a^{-1} k\, w_i^*$

$\Rightarrow g = k^{-1} a u^{-1} \in KAN$. The last step also gives $\det(k) = 1$.

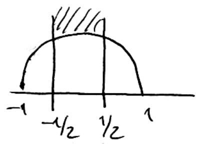
§7.2 Siegel sets

We now let go of the ass. that Γ is a lattice.

Recall (for purposes of inspiration): $SL(2,\mathbb{R}) \curvearrowright \mathbb{H}^2 = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$

by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \dfrac{az+b}{cz+d}$. Then we have a fund domain: (not a strict fund. dom.)

$$\overline{\mathcal{F}}_0 = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0,\ |z| \geq 1,\ -\tfrac{1}{2} < \operatorname{Re} z < \tfrac{1}{2}\}$$



Lemma. $H < G$ closed subgroup, $\overline{\mathcal{F}}$ strict fund domain for $\Gamma \curvearrowright G/H$.

Then $\forall x \in G/H$: $\mathcal{F} := \{g \in G \mid gx \in \overline{\mathcal{F}}\}$ is a strict fund domain for $\Gamma \curvearrowright G$.

Recall. $\Gamma \curvearrowright Y$ for a top space Y, $\Gamma \subset Y$ Borel, $\mathcal{F} \to Y/\Gamma$ is bijective means that $\mathcal{F}$ is a strict fund. domain.

PF OF LEMMA: Exc. 7.2.1.

We obtain a fund domain $\mathcal{F}_0 = \{g \in SL(2,\mathbb{R}) \mid g(i) \in \overline{\mathcal{F}}_0\}$

Problem: this has a way too complicated shape.

Solution: build a coarse fund domain.

$$\overline{\mathcal{F}} := \{x + iy \mid c_1 \leq x \leq c_2,\ c_3 \leq y\} \quad \text{box}$$

Def. Let $\Gamma \curvearrowright Y$, Y top space, properly discontinuous action. Then $\mathcal{F} \subset Y$ is called a <u>coarse fund domain</u> if (1) $\Gamma\mathcal{F} = Y$

and (2) $\{\gamma \in \Gamma \mid \gamma\mathcal{F} \cap \mathcal{F} \neq \emptyset\}$ is finite.

Set $\mathcal{F} := \{g \in G \mid g(i) \in \overline{\mathcal{F}}\}$, $\quad N_{c_1,c_2} := \left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \,\middle|\, c_1 \leq t \leq c_2\right\}$,

$A_{c_3} := \left\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \,\middle|\, e^{2t} \geq c_3\right\}$, $\quad K := SO(2)$.

Then $\mathcal{F} = N_{c_1,c_2} A_{c_3} K$, see Exc. 7.2.10, just computation.

Def. $G = SL(n,\mathbb{R})$. A Siegel set is a set of the form $S = \bar{N}\bar{A}K$ where

$$\bar{N} = N_{c_1,c_2} = \{ n \in N \mid \forall i,j: \ c_1 \leq u_{ij} \leq c_2 \}$$

$$\bar{A} = A_{c_3} = \{ a \in A \mid \forall i = 1,\dots,n-1: \ a_{ii} \geq c_3 a_{i+1,i+1} \}$$

Notation: $S_{c_1,c_2,c_3} = N_{c_1,c_2} A_{c_3} K$.

Prop. S_{c_1,c_2,c_3} has finite measure wrt the Haar measure on G.

Pf: Step 1. Description of Haar measure on G.

Write $a \in A$, $\psi_A : N \longrightarrow N$

$$n \longmapsto ana^{-1}$$

Then if du is the Haar measure on N then

$$\int_N f(ana^{-1}) \, du = \int_N f(n) \rho(a) \, du \qquad \forall f \in C_c(N).$$

where $\rho(a) = |\det (D\psi_a)|$

Uniqueness + unimodularity yield (Exc. 7.2.12):

$$\int_G f \, dg = \int_K \int_N \int_A f(kua) \, da \, du \, dk$$

where da, du, dk are the Haar measures on A, N, K resp.

$$= \int_K \int_A \int_N f(kan) \rho(a) \, du \, da \, dk$$

Computation (Exc. 7.2.13): $\rho(a) = \prod_{i<j} \dfrac{a_{j,j}}{a_{i,i}}$

Using that du is the Lebesgue measure in coordinates it follows (Exc. 7.2.14) that $S_{c_1c_2c_3}$ has finite measure.

§7.3. $\exists c_1 c_2 c_3$ for which S_{c_1,c_2,c_3} is a coarse fundamental domain

Thm. $G = SL(n,\mathbb{R})$, $\Gamma = SL(n,\mathbb{Z}) \Rightarrow G = \Gamma S_{0,1,1/2}$

Lemma. $\mathcal{L} \subseteq \mathbb{R}^n$ $\mathbb{Z}$-lattice. Then $\exists$ ordered basis of $\mathbb{R}^n$ $\{v_1,\dots,v_n\}$ s.t.

(1) $\{v_i\}_i$ are a generating set of $\mathcal{L}$ as an ab. gp.

(2) $\|\mathrm{proj}_i^{\perp} v_{i+1}\| \geq \frac{1}{2} \|\mathrm{proj}_{i-1}^{\perp} v_i\|$ for $i = 1,\dots,n-1$ where $\mathrm{proj}_i : \mathbb{R}^n \longrightarrow V_i^{\perp}$ is the projection to $V_i^{\perp} = \mathrm{Span}_{\mathbb{R}} \{v_1,\dots,v_i\}^{\perp}$

PF OF LEMMA:
The proof is similar to one from the beginning of this course.

Choose $v_1 \in \mathcal{L}\setminus\{0\}$ of minimal length.

Define $v_2, \dots, v_n$ inductively:

given $v_1, \dots, v_i$ choose $v_{i+1} \in \mathcal{L}$ s.t. • $\mathrm{proj}_i^{\perp} v_{i+1}$ is as short as possible

• v_{i+1} lin. indep. of $\{v_1, \dots, v_i\}$, i.e. $\mathrm{proj}_i^{\perp} v_{i+1} \neq 0$.

Goal: verify (1) and (2).

(1) $\forall i = 1, \dots, n \qquad \mathcal{L}_i := \mathrm{span}_{\mathbb{Z}} \{v_1, \dots, v_i\}$

$\lightning$ Suppose $v_1, \dots, v_n$ do not generate $\mathcal{L}$, i.e. $\mathcal{L}_n \neq \mathcal{L}$.

Then $\exists i \geq 1$ minimal s.t. $\mathcal{L}_{i+1} \neq \mathcal{L} \cap V_{i+1}$

$\Rightarrow \mathrm{proj}_i^{\perp} \mathcal{L}_{i+1} \subsetneq \mathrm{proj}_i^{\perp} (\mathcal{L} \cap V_{i+1})$ because $\mathcal{L} \cap V_i = \mathcal{L}_i$.

$\Rightarrow \exists v \in \mathcal{L} \cap V_{i+1}$ with $\mathrm{proj}_i^{\perp}(v_{i+1}) = k\, \mathrm{proj}_i^{\perp}(v)$ for some $k \geq 2$

because $\mathrm{proj}_i^{\perp}(\mathcal{L} \cap V_{i+1})$ is a cyclic group.

But this contradicts minimality of norms. $\lightning$

(2) Wma $i = 1$ for simplicity.

Let $v_2^* := \mathrm{proj}_1^{\perp} v_2$. $\Rightarrow \exists \alpha \in \mathbb{R}: \ v_2 = v_2^* + \alpha v_1$

Choose $k \in \mathbb{Z}: \ |\alpha - k| \leq \frac{1}{2}$.

$\lightning$ If $\|\mathrm{proj}_1^{\perp} v_2\| < \frac{1}{2} \|v_1\|$ then $\|v_2 - k v_1\| = \|v_2^* - (\alpha - k) v_1\|$

$\leq \|v_2^*\| + |\alpha - k|\, \|v_1\| < \|v_1\|$

which contradicts minimality of $\|v_1\|$. $\lightning$

05.07

PF OF THM: $G = \mathfrak{S}^{-1}_{0,1,1/2}\, \Gamma$ where $\mathfrak{S}^{-1}_{0,1,\frac{1}{2}} = K A^-_{1/2} N_{0,1} \Gamma$

and $A^-_c := \{a^{-1} \mid a \in A_c\} = \left\{ \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \,\middle|\, a_{ii} \leq \frac{a_{i+1,i+1}}{c} \ \ \forall i = 1, \dots, n-1 \right\}$

Goal: given $g \in G$ show $\exists \gamma \in SL(n, \mathbb{Z})$: $\ g\gamma \in K A^-_{1/2} N_{0,1}$

Let $\mathcal{L} := g\mathbb{Z}^n$ and $\varepsilon_1, \dots, \varepsilon_n$ the std basis of $\mathbb{R}^n$.

Lemma $\Rightarrow$ $v_1, \dots, v_n$ generating set

Claim: wma $g\varepsilon_i = v_i$, i.e. $\exists \gamma \in \Gamma$: $g\gamma\varepsilon_i = v_i$

Pf: Any pair of generator sets for $\mathbb{Z}^n$ are related by a $GL(n,\mathbb{Z})$-matrix: $\exists \gamma \in GL(n,\mathbb{Z})$: $g\gamma\varepsilon_i = v_i$

By the same argument as in the pf of Mahler compactness:

γ preserves volume hence lies in $SL(n,\mathbb{Z})$. ✓

Now use Iwasawa decomposition $g = kau$

rotate such that $k = e$ $\Rightarrow$ $g = au$, hence upper triangular with diagonal entries coming from a.

Step 1. $a \in A_{1/2}$, $a_{ii} = g_{i,i} = \| \text{proj}^{\perp}_{i-1} \, g\varepsilon_i \| = \| \text{proj}^{\perp}_{i-1} \, v_i \|$

$\Rightarrow a_{ii} \leq \frac{1}{2} a_{i+1,i+1} \Rightarrow a \in A^-_{1/2}$

Step 2. Using another $SL(n,\mathbb{Z})$-matrix.

Claim. $N = N_{0,1} N_{\mathbb{Z}}$

E.g. in dim 2: $\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{\in N_{\mathbb{Z}}} \cdot \underbrace{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}}_{\in N} = \begin{pmatrix} 1 & x+1 \\ 0 & 1 \end{pmatrix}$ □

Ch. 8. Real rank

Def. A closed connected subgroup $T < G$ is a torus if it is diagonalisable over $\mathbb{C}$ i.e. $\exists g \in GL(n,\mathbb{C})$ s.t. gTg^{-1} consists of diagonal matrices.

A torus is called $\mathbb{R}$-split if it is diagonalisable over $\mathbb{R}$.

Ex. 1) $A = \left\{ \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \right\}^0 \subset SL(n,\mathbb{R})$ is an $\mathbb{R}$-split torus

2) $SO(1,1)^0 \subset SL(2,\mathbb{R})$ is an $\mathbb{R}$-split torus (Exc. 8.1.1)

3) $SO(2) < SL(2,\mathbb{R})$ is a torus but not $\mathbb{R}$-split

$\quad \|$

$\{A \in SL(2,\mathbb{R}) \mid AA^T = Id\}$

Thm. $A_1, A_2 < G$ maximal $\mathbb{R}$-split tori $\Rightarrow$ $\exists g \in G:$ $gA_1g^{-1} = A_2$

Nb. This theorem uses the semisimplicity of G and does not hold in general.

Def. $\text{rank}_{\mathbb{R}}(G) :=$ dimension of (a) maximal $\mathbb{R}$-split torus in G.

Ex. 1) $\text{rank}_{\mathbb{R}} SL(n,\mathbb{R}) = n-1$. Minor ignored technicality: what if we take a different embedding? One can show that this doesn't matter.

2) $\text{rank}_{\mathbb{R}} SL(n,\mathbb{C}) = \text{rank}\ SL(n,\mathbb{H}) = n-1$

This is because the only matrices that remain diagonal under the embeddings $SL(n,\mathbb{C}) \hookrightarrow SL(2n,\mathbb{R})$ resp. $SL(n,\mathbb{H}) \hookrightarrow SL(4n,\mathbb{R})$ are real-valued diagonal matrices.

3) $\text{rank}_{\mathbb{R}} G = 0 \iff G$ is compact.

Prop. $\text{rank}_{\mathbb{R}} SO(m,n) = \min(m,n)$. In particular, $\text{rank}_{\mathbb{R}} SO(1,n) = 1$.

Pf: $\left(SO(1,1)^{\min(m,n)}\right)^0 < SO(m,n)$

$\dim SO(1,1)^{\min(m,n)} = \min(m,n) \Rightarrow \text{rank}_{\mathbb{R}} SO(m,n) \geqq \min(m,n)$

NTS reversed inequality. (Coward's approach)

Let A be a maximal $\mathbb{R}$-split torus in $SO(m,n)$.

$a \in A \setminus \{e\}$ since a is $\mathbb{R}$-diagonalisable and nontrivial

$\Rightarrow \exists v \in \mathbb{R}^n \setminus \{0\},\ \exists \lambda \in \mathbb{R} \setminus \{1\}$ s.t. $av = \lambda v$.

Take $a \approx e \rightarrow \lambda \neq -1$

$$\langle v, v \rangle_{m,n} = \langle av, av \rangle_{m,n} = \lambda^2 \langle v, v \rangle_{m,n}$$

Since $\lambda^2 \neq 1$: $\langle v, v \rangle_{m,n} = 0$

By the same argument: if $\dim(A) \geq k$ we get a totally isotropic subspace, (i.e. $V \subset \mathbb{R}^{m+n}$, $\langle v, v \rangle_{m,n} = 0 \ \forall v \in V$) of dim k.

Since $\max \left\{ \dim V, \middle|\ V \subset (\mathbb{R}^{m+n}, \langle \cdot, \cdot \rangle_{m,n}) \text{ tot. isotr subsp.} \right\} = \min(m,n)$,

$\operatorname{rank}_{\mathbb{R}} SO(m,n) \leq \min(m,n)$ as desired. □

↓ NOT IN THE EXAM

Ch 16. Margulis superrigidity theorem

Thm. Suppose $\operatorname{rank}_{\mathbb{R}} G \geqslant 2$. and G is connected,

- $\Gamma < G$ irreducible
- H connected non-compact simple linear Lie group with trivial center
- $\varphi : \Gamma \rightarrow H$ homomorphism
- $\varphi(\Gamma) \subset H$ is Zariski dense

Then φ extends to a continuous homomorphism $\hat{\varphi} : G \rightarrow H$.

NB. This is not the most general version.

Applications (§ 16.2)

Thm. (Mostow rigidity) Suppose

- G_1, G_2 are connected with trivial center and no compact factors
- $PSL(2,\mathbb{R}) \not\cong G$
- $\Gamma_i < G_i$ irreducible lattices ($i = 1,2$)
- $\varphi : \Gamma_1 \rightarrow \Gamma_2$ a group iso

Then φ extends to a cont iso $\hat{\varphi} : G_1 \rightarrow G_2$.

Rmk. For rank 1 cases, e.g. for $SO(1,n)$ with $n \geq 3$, a different approach is needed.

Flat vector bundles

Def. $\varphi : \Gamma \longrightarrow GL(n,\mathbb{R})$ a homomorphism. One obtains an action $\Gamma \curvearrowright G \times \mathbb{R}^n$:

$$(x, v)\gamma = (x\gamma, \varphi(\gamma^{-1})v)$$

Let $\mathcal{E}_\varphi = G \times \mathbb{R}^n / \Gamma$. Then there is a map $\pi : \mathcal{E}_\varphi \longrightarrow G/\Gamma$ turning $\mathcal{E}_\varphi$ into a vector bundle with fiber $\mathbb{R}^n$. This is called a flat vector bundle

We formulate a variation of Margulis' theorem:

Thm. Assume

- $G = SL(n,\mathbb{R})$
- $\Gamma < G$ s.t. G/Γ is not compact
- $\varphi : \Gamma \longrightarrow GL(n,\mathbb{R})$ any homomorphism.

Then there are

- $\hat{\varphi} : G \longrightarrow GL(n,\mathbb{R})$ cont. homomorphism
- $\Gamma' < \Gamma$ finite index subgp.

for which $\varphi(\gamma) = \hat{\varphi}(\gamma) \quad \forall \gamma \in \Gamma'$

Prop. $G = SL(n,\mathbb{R})$, $\Gamma = SL(n,\mathbb{Z})$. If $\mathcal{E}_\varphi$ is any flat vector bundle over G/Γ then $\exists \Gamma' < \Gamma$ is of finite index such that the lift $\mathcal{E}_{\varphi}'$ of $\mathcal{E}_\varphi$ to the finite cover $G/\Gamma' \longrightarrow G/\Gamma$ is trivial, i.e. isometric to $G/\Gamma' \times \mathbb{R}^n$.

PF OF PROP: Take Γ' as in the Theorem.

Define $T : G \times \mathbb{R}^n \longrightarrow G \times \mathbb{R}^n$

$$(g, v) \longmapsto (g, \hat{\varphi}(g)v)$$

$\forall \gamma \in \Gamma'$: $T((g,v)\gamma) = (g\gamma, \hat{\varphi}(g)v) \Rightarrow T$ factors through the bundle isometry $\mathcal{E}_{\varphi}' \simeq G/\Gamma' \times \mathbb{R}^n$.

§ 16.3 Superrigidity ⇒ arithmeticity

Thm. (Margulis) $\mathrm{rank}_{\mathbb{R}} G \geq 2$, $\Gamma < G$ irreducible lattice ⇒ Γ is arithmetic.

10.07.2018

Pf (SKETCH): Goal: show that we can assume $\Gamma < G_{\mathbb{Z}}$. Then we automatically get that Γ is commensurable to $G_{\mathbb{Z}}$, hence Γ is arithmetic.

Step 1. Superrigidity ⇒ the matrix entries of elements of Γ are algebraic.

Step 2. Use restriction of scalars to make them rational.

Step 3. Superrigidity over $\mathbb{Q}_p$ ⇒ $\forall p$ prime $\exists N_p$ universal upper bound, i.e. no matrix entry of any elt of Γ has a denominator divisible by a higher power of p. ("Denominators don't get too bad.")

Step 4. ⇒ $\exists \Gamma' < \Gamma$ finite index, contained in $G_{\mathbb{Z}}$.

Now we do these steps in detail.

Step 1. $\lightning$ Suppose $\exists \gamma \in \Gamma$ $\exists i,j$ s.t. γ_{ij} is transcendental.

Galois thy ⇒ $\forall \alpha \in \mathbb{R} \setminus \overline{\mathbb{Q}}$ $\exists \phi \in \mathrm{Aut}(\mathbb{C}) : \phi(\gamma_{ij}) = \alpha$

We obtain an automorphism $\tilde{\phi} \in \mathrm{Aut}\, SL(l, \mathbb{C})$ by applying ϕ to the entries.

⇒ get a homomorphism $\rho := \tilde{\phi}|_{\Gamma} : \Gamma \longrightarrow SL(l, \mathbb{C})$

Superrigidity ⇒ $\exists \varphi : G \longrightarrow SL(l, \mathbb{C})$ cont. homomorphism,

$\hat{\varphi} = \varphi$ on a finite index subgp of Γ

⇒ get uncountably many distinct homomorphisms $G \to SL(l, \mathbb{C})$ by varying α.

Lie thy + some extra argument ⇒ $\lightning$

Step 2. Γ finitely generated ⇒ the subfield of $\mathbb{C}$ generated by the matrix entries is a finite extension of $\mathbb{Q}$.

Apply restriction of scalars.

Step 3. We need the following.

Thm. (Margulis superrigidity / $\mathbb{Q}_p$) G connected, $\mathrm{rank}_{\mathbb{R}} G \geq 2$,

- Γ irreducible,
- $\Gamma \xrightarrow{\varphi} SL(n, \mathbb{Q}_p)$ homomorphism

Then $\overline{\varphi(\Gamma)}$ is compact, i.e. $\exists N \in \mathbb{Z}$ s.t. $\forall$ matrix entry of $\forall$ elt of $\varphi(\Gamma)$ is in $p^N \mathbb{Z}_p$.

Since $\mathbb{Q} \subset \mathbb{Q}_p$, this completes Step 3.

Step 4. Let $D := \{$ denominators of matrix entries of $\Gamma \} \subseteq \mathbb{N}$.

We know that for a given p, D only contains finitely many.

Wts D is bounded, i.e. only finitely many primes contribute to D.

Γ is finitely generated, let $p_1, \dots, p_r$ be the primes appearing in its generators. Then every elt of D is of the form $p_1^{m_1} \dots p_r^{m_r}$ with $m_i \in \mathbb{N}$.

By Step 3, the m_i are bounded $\Rightarrow \exists M \in \mathbb{N}: \Gamma \subseteq \frac{1}{M} \mathrm{Mat}_{\ell \times \ell}(\mathbb{Z})$

$\Rightarrow \exists$ fin. index $\Gamma' < \Gamma$ lying in $SL(\ell, \mathbb{Z})$. by Exc. 16.3.2.

The following has a similar proof:

Thm. (Commensurability criterion for arithmeticity)

Assume G to be connected and Γ to be irreducible.

Then Γ is arithmetic $\Leftrightarrow$ $\mathrm{Comm}_G(\Gamma)$ is dense in G.

(topologically, not Zariski)

Lemma. $\varphi: \Gamma \longrightarrow GL(n, \mathbb{R})$ extends to a continuous hom. $\tilde{\varphi}: G \longrightarrow GL(n, \mathbb{R})$ iff $\exists V \subseteq \mathrm{Sect}(E_\varphi)$ G-invariant subspace s.t. $V \longrightarrow V_{[\varphi]}$ is bijective where $[\varphi] \in G/\Gamma$ is the img of φ in G/Γ.

Pf: "$\Leftarrow$" G-invariance of $V \Rightarrow$ get map $G \longrightarrow GL(V)$

$V_{[\varphi]} = \mathbb{R}^n \Rightarrow V = V_{[\varphi]} = \mathbb{R}^n \Rightarrow$ get $G \longrightarrow GL(n, \mathbb{R})$

To see that this extends φ note that $\mathrm{Sect}(E_\varphi) \overset{1:1}{\longleftrightarrow}$ (right Γ-equivariant maps $G \longrightarrow \mathbb{R}^n$)

If $\xi: G \longrightarrow \mathbb{R}^n$ is right Γ-equivariant, i.e. $\xi(g\gamma) = \varphi(\gamma^{-1})\xi(g)$

then $\bar{\xi}: G/\Gamma \longrightarrow E_\varphi$ defined by $\bar{\xi}(g\Gamma) := [(g, \xi(g))]$ is a well-def'd section.

"$\Rightarrow$" $v \in \mathbb{R}^n$, $g \in G$, define $\xi_v(g) := \tilde{\varphi}(g^{-1}) \cdot v$

This is a right Γ-equivariant map $\rightarrow$ defines a section.

$V := \{\xi_v \mid v \in \mathbb{R}^n\}$, $v \longmapsto \xi_v$ is a linear G-equivariant map

$\Rightarrow$ V is a G-invariant subspace

Since $\xi_v(e) = \tilde{\varphi}(e)v = v$, we have the desired bijection. $\square$

Now we look at where higher rank comes into play.

<u>Lemma.</u> If $\operatorname{rank}_{\mathbb{R}} G \geq 2$ then $\exists r \in \mathbb{N}$ $\exists L_1, \dots, L_r < G$ closed subgroups s.t.

1) $G = L_r L_{r-1} \dots L_1$

2) both $H_i := L_i \cap A$, $H_i^{\perp} := C_A(L_i)$ are non-compact.

Pf: Lie groups. (Maybe next semester?) $\square$

<u>Lemma.</u> If we have

- $H < A$ closed non-compact subgroup
- $V \subset \operatorname{Sect}(E_\varphi)$ H-invariant finite dimensional subspace

$\Rightarrow$ $\langle C_G(H), V \rangle$ has finite dimension. $\square$

This lemma also goes into the pf of superrigidity.

PROOF OF SUPERRIGIDITY:

<u>Step 1.</u> Find a nonzero section $\sigma : G/_\Gamma \longrightarrow E_\varphi$ which is A-invariant.

Set $V_0 := \mathbb{R}_\sigma \subset \operatorname{Sect} E_\varphi$

Since $H_1 = L_1 \cap A < A$, V_0 is H_1-invariant

Using $H_1^{\perp}$ and the prev lemma we get a new fin dim subspace

of $\operatorname{Sect}(E_\varphi)$, invariant under $A, H_1, H_1^{\perp}$

Repeat $\rightarrow$ get invariant subspace for G-action which is good enough.

Recall: sections of $E_\varphi \overset{1:1}{\longleftrightarrow}$ (right Γ-equivariant maps $G \longrightarrow \mathbb{R}^n$)

<u>Fact</u> (§16.6) For some embedding $H \hookrightarrow SL(\ell, \mathbb{R})$:

1) H acts irreducibly on $\mathbb{R}^\ell$

2) $\exists \xi : G/_A \longrightarrow \mathbb{R}^n$ Γ-equivariant function, measurable, $\xi \neq 0$.

Something is amenable, use ergodic theory $\rightsquigarrow$ $\square$